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Variational analysis of the deconfinement phase transition

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Abstract: We study the deconfining phase transition in 3+1 dimensional pure SU(N) Yang-Mills theory using a gauge invariant variational calculation. We generalize the variational ansatz of [9] to mixed states (density matrices) and minimize the free energy. For $N \geq 3$ we find a first order phase transition with the transition temperature of $\mathsf{T_c} \simeq 450\,\mathrm{MeV}$. Below $\mathsf{T_c}$ the Polyakov loop has vanishing expectation value, while above $\mathsf{T_c}$, its average value is nonzero. According to the standard lore this corresponds to the deconfining transition. Within the accuracy of our approximation the entropy of the system in the low temperature phase vanishes. The latent heat is not small but, rather, is of the order of the nonperturbative vacuum energy.

Keywords: Nonperturbative Effects, Confinement, QCD.

Contents

1. Introduction

Attempts to understand the nature of the deconfining phase transition in QCD date back almost 25 years. Since the pioneering work of Polyakov [1], much effort has been made to study the basic physics as well as the quantitative characteristics of the transition. The high temperature phase of QCD is widely believed to resemble an almost free plasma of quarks and gluons. At asymptotically high temperatures this is confirmed by explicit perturbative calculations of the free energy [2]. Perturbation theory in its simplest form, however, is valid only at unrealistically high temperatures. In recent years a different promising avenue has been explored. This incorporates analytical resummation of the effects of the gluon screening mass into the 3D effective lagrangian, which is then solved numerically by 3D lattice gauge theory methods [3, 4]. The results of this approach seem to be in good agreement [6] with direct 4D lattice gauge theory calculations [5] all the way down to $2T_c$. The numerical results indicate that, interestingly enough, although asymptotically the free energy does approach that of the free partonic plasma, the deviations from the Stefan-Boltzmann law even at temperatures of order $10T_c$ are quite sizable, of order of 15%. One can interpret this as an indication that the interesting physics of the transition region remains important even at these high temperatures.

Although we are quite advanced in the understanding of the high temperature phase, the transition region itself is understood very poorly. This region of temperatures $T_c < T < 2T_c$, is of course the most interesting one, since it is in this region that the transition between "hadronic" and "partonic" degrees of freedom occurs. In fact, it is quite possible that the dynamics of the transition region is dominated by the Z_N magnetic vortices [7],

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which are responsible for the abrupt increase in entropy [8]. The study of the transition region is a complicated and inherently nonperturbative problem which has not been tackled so far by analytical methods.

The purpose of this paper is to explore an analytical investigation of the phase transition region. Since the nature of the problem is nonperturbative, a nonperturbative method is called for. Unfortunately there are not many of those on the market. It does however seem to us that the variational method developed several years ago in [9] is quite suitable. We are encouraged by the fact that the variational approximation of [9] manages to reproduce all known infrared features in lower dimensional confining theories [12]–[15], and also gives a stable picture of instanton dynamics in QCD [16]. Even though, as any variational method, it has an element of guesswork about it, it is a first principle calculation. For such a difficult physics problem as the one at hand, clearly even an exploratory investigation which provides a qualitative picture should be very valuable and is well worth attempting.

In this paper, therefore, we will study the deconfining phase transition in a pure SU(N) Yang-Mills theory using the variational approach of [9] suitably extended to finite temperature. We will minimize the relevant thermodynamic potential at finite temperature, i.e. the Helmholtz free energy, on a set of gauge invariant density matrices.

The results of our calculation are quite promising. We find that the phase transition in gluodynamics is closely related to the symmetry breaking phase transition in the effective σ-model for the Polyakov loop variable U. This σ-model arises explicitly in our calculation, and its parameters depend on the variational parameters of the trial density matrix. At low temperature the effective σ-model is in the disordered phase, and $\langle U \rangle = 0$. The entropy of the Yang-Mills thermal density matrix vanishes in the low temperature phase. At the critical temperature of order $\mathsf{T_c} \simeq 450\,\mathrm{MeV}$, the best variational density matrix changes abruptly, and above $\mathsf{T_c}$ corresponds to the ordered phase of the σ-model. Thus at $\mathsf{T} > \mathsf{T_c}$ we find $\langle U \rangle \neq 0$. The transition itself turns out to be strongly first order for large-N. This is consistent with recent lattice gauge theory calculations [10]. These results are not surprising in themselves. They are expected from the general relation between the average value of the Polyakov loop and the (de)confining properties of the density matrix. Moreover, the order of the transition is predicted by the Svetitsky-Yaffe conjecture [11]. The vanishing of the entropy in the confining phase is also very natural, since the glueballs are heavy and their contribution to the entropy should be suppressed by the Boltzmann factor $\exp[-M_g/T]$. Nevertheless, we are not aware of any first principle analytic calculation in QCD which provides an explicit realization of these general arguments.

This paper is structured as follows. In section 2 we recap the variational ansatz of [9] for energy minimization at zero temperature and recall the methods of evaluating the expectation values in the variational wave function as well as the results of the minimization procedure. In section 3 we generalize the variational ansatz so that it incorporates not only pure states but also general density matrices. We introduce the two variational parameters with respect to which the free energy is to be minimized. In section 4 we recast the problem of the minimization of free energy in terms of an effective nonlinear σ-model. In section 5 we calculate the Helmholtz free energy of the trial density matrix and perform the minimization. We conclude in section 6 by discussing our results.

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2. The variational calculation at zero temperature

The variational method proposed in [9] is based on minimization of the expectation value of the hamiltonian on a set of gauge invariant states. The set of states is chosen so that their functional form is a relatively simple generalization of the vacuum of the free theory. Ideally, one would like to choose a set of variational states that is both simple enough to perform explicit calculations and rich enough to allow for variations of its parameters to span interesting physics. One starts then with a gaussian state

$$\Psi_0[A_i^a] = \exp\left\{-\frac{1}{2}\int d^3x d^3y A_i^a(x)\left(G^{-1}\right)_{ij}^{ab}(x,y)A_j^b(y)\right\}, \tag{2.1}$$

where the set of functions $G_{ab}^{ij}(x)$ are variational parameters. Expectation values of various operators are easily calculable in this set of states. One also expects that the freedom allowed by variation of G is wide enough to probe the nonperturbative physics of confinement. The problem is that these states are not gauge invariant and, therefore, as such do not belong to the physical Hilbert space of gluodynamics. To remedy this problem one projects the states onto the gauge invariant subspace by gauge transforming them and integrating over the whole gauge group:

$$\Psi[A_i^a] = \int DU(x)\exp\left\{-\frac{1}{2}\int d^3x d^3y A_i^{Ua}(x)G_{ij}^{-1ab}(x-y)\ A_j^{Ub}(y)\right\} \tag{2.2}$$

with A_i^{Ua} defined as

$$A_i^{Ua}(x) = S^{ab}(x)A_i^b(x) + \lambda_i^a(x), \tag{2.3}$$

and

$$S^{ab}(x) = \frac{1}{2}\,\mathrm{tr}\left(\tau^a U^\dagger \tau^b U\right); \qquad \lambda_i^a(x) = \frac{i}{g}\,\mathrm{tr}\left(\tau^a U^\dagger \partial_i U\right) \tag{2.4}$$

with τ — traceless $N\times N$ hermitian matrices — normalized by

$$\mathrm{tr}\left(\tau^a\tau^b\right) = 2\delta^{ab}, \tag{2.5}$$

and obeying the completeness condition for SU(N)

$$\tau_{ij}^a\tau_{kl}^a = 2\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \tag{2.6}$$

The integration in eq. (2.2) is performed over the space of special unitary matrices with the SU(N) group invariant measure. This integration projects the original gaussian state onto a colour singlet. Due to the projection operation, the calculation of expectation values in this state is much more involved than in the case of a simple gaussian. Thus it is a very difficult task to perform the full functional minimization with respect to $G_{ab}^{ij}(x)$. This can be done and was indeed performed in simpler calculations in 2+1 dimensional models [12, 13]. In these models the approach was also extended to provide for the calculation of the confining potential between the external charges [14]. It has also been applied

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to 1+1 dimensional models with fermions [15]. In these simple models the variational approximation has performed very well, reproducing all known results.

In nonabelian theories, unfortunately, full functional minimization is beyond our calculational abilities. Therefore to make headway, the width of the gaussian G was restricted in [9] to a one parameter form as follows

$$G^{ab}_{ij}(x-y) = \delta^{ab}\delta_{ij}G(x-y) \tag{2.7}$$

with the Fourier transform of G taken to be

$$G^{-1}(k) = \begin{cases} \sqrt{k^2} & \text{if } k^2 > M^2 \\ M & \text{if } k^2 < M^2 \end{cases} . \tag{2.8}$$

The dimensional parameter M is thus the only variational parameter in the calculation.

The expectation value of any gauge invariant operator O in the variational state eq. (2.2) is given by the following integral

$$\begin{aligned} \langle O\rangle &= \frac{1}{Z}\int DU\langle O\rangle_A\,, \\ \langle O\rangle_A &= \int DAe^{-\frac{1}{2}\int dxdy A^{Ua}_i(x)G^{-1}(x-y)A^{Ua}_i(y)}Oe^{-\frac{1}{2}\int dx'dy' A^b_j(x')G^{-1}(x'-y')A^b_j(y')}\,. \end{aligned} \tag{2.9}$$

Note that only one group integral is present. This is because for a gauge invariant operator O only one of the states has to be gauge projected.

Since the gauge transform of a vector potential is a linear function of A eq. (2.3), for fixed $U(x)$ this is a gaussian integral, and can therefore be performed explicitly for any reasonable operator O. The nontrivial part of the calculation is the path integral over the group variable $U(x)$. Consider first the normalization factor Z. After integrating over the vector potential we obtain:

$$Z = \int DU\exp\{-\mathcal{S}[U]\} \tag{2.10}$$

with the action

$$\mathcal{S}[U] = \frac{1}{2}\,\mathrm{Tr}\ln\mathcal{M} + \frac{1}{2}\lambda\left[G + SGS^T\right]^{-1}\lambda\,. \tag{2.11}$$

Here

$$S^{ab}_{ij}(x,y) = S^{ab}(x)\delta_{ij}\delta(x-y)\,, \qquad \mathcal{M}^{ab}_{ij}(x,y) = \left[S^{Tac}(x)S^{cb}(y) + \delta^{ab}\right]G^{-1}(x-y)\delta_{ij}\,, \tag{2.12}$$

and we use matrix product notation, which implies summation over the colour and Lorentz indices, as well as integration over the spatial coordinates. In the rest of this paper Tr stands for summation over all indices, including the space coordinates, while tr stands for summation over the colour indices alone. The path integral eq. (2.10) defines a partition function of a nonlinear σ-model with the target space $\mathrm{SU}(N)/Z_N$ in three dimensional euclidean space. The fact that the target space is $\mathrm{SU}(N)/Z_N$ rather than $\mathrm{SU}(N)$, follows from the observation that the action eq. (2.11) is invariant under local transformations belonging to the centre of $\mathrm{SU}(N)$. This can be trivially traced back to invariance of A^a_i under gauge transformations that belong to the centre of the gauge group.

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The matrix $U(x)$ has a well defined gauge invariant meaning. It arises in eq. (2.9) as a relative gauge transformation between the bra and the ket gaussian wave functionals. The contribution of a given $U(x)$ to the partition function eq. (2.10) and to other expectation values corresponds to the contribution due to the off diagonal matrix element between the initial gaussian and the gaussian gauge rotated by $U(x)$. Therefore, if matrices $U(x)$ which are far from unity give a significant contribution to the partition function, it means that the off diagonal contributions are large and gauge projection is indeed mandatory to get physically sensible results.

The action of this σ-model is rather complicated. It is a nonlocal and a nonpolynomial functional of $U(x)$. The following set of approximations was employed in [9] to estimate this path integral. First, since the bare coupling constant of the Yang Mills theory is small and it also enters as a coupling constant of the nonlinear σ-model, the high momentum modes of the field U are integrated out perturbatively. The result of the perturbative one loop integration of the modes with spatial momenta $k > M$ is a much simpler σ-model which involves low momentum modes only. The action of this σ-model is

$$\mathcal{S}_L[U] = \frac{M}{2g^2(M)} \operatorname{tr} \int d^3x \; \partial_i U^\dagger(x) \partial_i U(x) \,, \tag{2.13}$$

where M also serves as the ultraviolet cutoff. To be more precise, due to the Z_N local symmetry of the original theory eq. (2.11), the action for the low momentum modes is slightly different. The derivatives should be understood as Z_N covariant derivatives. The most convenient way to write this action, would be to understand $U(x)$ as belonging to $\mathrm{U}(N)$ rather than $\mathrm{SU}(N)$, and to introduce a $\mathrm{U}(1)$ gauge field by

$$\mathcal{S}_L = \frac{1}{2}\frac{M}{g^2} \operatorname{tr} \int d^3x \; (\partial_i - iA_i) U^\dagger(x) (\partial_i + iA_i) U(x) \,. \tag{2.14}$$

This defines a σ-model on the target space $\mathrm{U}(N)/\,\mathrm{U}(1)$, which is isomorphic to $\mathrm{SU}(N)/Z_N$. However, since the difference between the two actions is not important at large number of colours, we will disregard this subtlety in the following and use eq. (2.13).

As a result of the running of the QCD coupling constant, the effective coupling $g^2(M)$ is given by the one loop QCD formula [17]. Clearly for the perturbative integration to be consistent we should find that the energy is eventually minimized for high enough value of the scale M, so that $\alpha_s(M) \ll 1$. The actual value found in [9] is $\alpha_s(M) \simeq .25$, which is indeed reasonably small. To solve the resulting low energy σ-model the mean field approximation and perturbation theory are used. We will describe this procedure here in some detail, since we will use it in later sections to analyze the finite temperature case.

The hamiltonian of the pure Yang-Mills theory is

$$\mathsf{H} = \int d^3x \left[\frac{1}{2} E_i^{a2} + \frac{1}{2} B_i^{a2} \right] , \tag{2.15}$$

where

$$\begin{aligned} E_i^a(x) &= i \frac{\delta}{\delta A_i^a(x)} \,, \\ B_i^a(x) &= \frac{1}{2} \epsilon_{ijk} \left\{ \partial_j A_k^a(x) - \partial_k A_j^a(x) + g f^{abc} A_j^b(x) A_k^c(x) \right\} . \end{aligned} \tag{2.16}$$

To calculate the expectation value of H in the variational state we first perform the integral over the gauge fields A at fixed value of the matrix U. The result of this integration (for details see [9]) for the energy density is

$$
\begin{aligned}
\frac{\langle 2\mathsf{H}\rangle}{V} &= \frac{3N^2}{2}G^{-1}(x,x) + N^2\partial_i^x\partial_i^y G(x-y)|_{x=y} - \\
&- \frac{1}{4V}\int d^3x d^3y \langle \lambda^a_{iH}(x) G^{-2}(x-y)\lambda^a_{iH}(y)\rangle_U + \frac{1}{4}\langle(\epsilon_{ijk}\partial_j\lambda^a_{kH})^2\rangle_U - \\
&- \frac{M^2}{4V}\int d^3x \langle\lambda^a_{iL}(x)\lambda^a_{iL}(x)\rangle_U\,, \qquad (2.17)
\end{aligned}
$$

where the averaging over the U field should be performed with the σ-model action, eq. (2.11). As the high momentum modes are only considered to one loop order, the σ-model action reduces to

$$
\mathcal{S} = \frac{1}{4}\int dxdy\lambda^a_{iH}(x)G^{-1}(x-y)\lambda^a_{iH}(y) + \frac{M}{2g^2(M)}\operatorname{tr}\int d^3x\ \partial_i U_L^\dagger(x)\partial_i U_L(x)\,. \qquad (2.18)
$$

The subscripts H and L indicate that the corresponding expressions contain only high and low momentum components of the fields respectively. Integration over the high momentum modes gives

$$
\frac{2\langle\mathsf{H}\rangle}{V} = \frac{N^2}{10\pi^2}M^4 - \frac{M^2}{4V}\int d^3x\langle\lambda^a_{iL}(x)\lambda^a_{iL}(x)\rangle_U\,, \qquad (2.19)
$$

where now only the contribution of the low momentum modes remains to be evaluated.

This last step of the calculation is the most difficult one. The nonlinear σ-model eq. (2.13) possesses a global $SU_L(N)\otimes SU_R(N)$ symmetry. Thus it has a symmetry restoring phase transition as a function of M. The coupling constant $g^2(M)$ enters this action in the same way as temperature enters in the classical statistical problem of the SU(N) spin system. High values of M correspond to low values of $g^2(M)$ and thus to the low temperature regime in the σ-model. In this regime the global $SU_L(N)\otimes SU_R(N)$ symmetry is spontaneously broken down to the vector subgroup $SU_V(N)$. The model itself is weakly interacting and thus one can use perturbation theory. However, it is well known that the model undergoes a phase transition from the ordered low temperature phase to the disordered phase at some critical temperature g_c^2. The disordered phase is clearly nonperturbative. To treat it quantitatively the mean field approximation was used in [9]. The partition function of the σ-model is rewritten by introducing a (hermitian matrix) auxiliary field α which imposes a unitarity constraint on $U(x)$

$$
\begin{aligned}
Z &= \int DUD\alpha\exp\left(-\mathcal{S}\left[U,\alpha\right]\right)\,, \\
\mathcal{S}[U,\alpha] &= \frac{M}{2g^2(M)}\operatorname{tr}\int d^3x\left[\partial_i U^\dagger(x)\partial_i U(x) + \alpha\left(U^\dagger U - 1\right)\right]. \qquad (2.20)
\end{aligned}
$$

In this step we have disregarded the fact that U is an SU(N) matrix rather than U(N). Thus, strictly speaking, our calculation applies only at large-N. The mean field equations for this action are

$$
\langle U^\dagger U\rangle = 1\,, \qquad (2.21)
$$

$$
\langle\alpha U\rangle = 0\,. \qquad (2.22)
$$

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From eq. (2.22) it follows that either $\langle\alpha\rangle = 0$, $\langle U\rangle \neq 0$ (the ordered, broken symmetry phase with massless Goldstone bosons), or $\langle\alpha\rangle \neq 0$, $\langle U\rangle = 0$ (the disordered, unbroken phase with massive excitations).

In the disordered phase the expectation value of α is proportional to the unit matrix

$$\langle\alpha_{\alpha\beta}\rangle = \alpha^2 1_{\alpha\beta}\,. \tag{2.23}$$

Eq. (2.21) then becomes

$$2N^2\frac{g^2(M)}{M}\int_0^M \frac{d^3k}{(2\pi)^3}\frac{1}{k^2+\alpha^2} = \frac{N^2 g^2(M)}{\pi^2}\left(1-\frac{\alpha}{M}\arctan\frac{M}{\alpha}\right) = N\,. \tag{2.24}$$

The gap equation, eq. (2.24), has a solution only for couplings $g^2(M)$ larger than the critical coupling g_c^2, which is determined by the condition that $\alpha = 0$

$$\alpha_s^c = \frac{g_c^2}{4\pi} = \frac{\pi}{4}\frac{1}{N}\,. \tag{2.25}$$

The low momentum mode contribution to the ground state energy is

$$N^2 M\int_0^M \frac{d^3k}{(2\pi)^3}\frac{k^2}{k^2+\alpha^2} = \frac{N^2}{2\pi^2}M\left[\frac{1}{3}M^3 - \alpha^2 M + \alpha^3\arctan\frac{M}{\alpha}\right]. \tag{2.26}$$

The final mean field expression for the ground state energy density is

$$\mathsf{E} = \frac{N^2}{4\pi^2}M^4\left[-\frac{2}{15}+\frac{\alpha^2}{M^2}\frac{\alpha_s^c}{\alpha_s(M)}\right], \tag{2.27}$$

where $\alpha_s(M)$ is the QCD coupling at the scale M, α_s^c is given by eq. (2.25), and α is determined by

$$\frac{\alpha}{M}\arctan\frac{M}{\alpha} = \frac{\alpha_s(M)-\alpha_s^c}{\alpha_s(M)}\,. \tag{2.28}$$

This function has the minimum at the critical point $\alpha_s(M) = \alpha_s^c$. With the one loop Yang-Mills β function and $\Lambda_{QCD} = 150\,\mathrm{MeV}$, we find for $N = 3$

$$M_c = \Lambda_{QCD}e^{\frac{24}{11}} = 8.86\Lambda_{QCD} = 1.33\,\mathrm{GeV}\,. \tag{2.29}$$

Eq. (2.27) is valid only in the disordered phase. Here we expect the mean field approximation to be reasonably good, since the spectrum of the σ-model is massive and fluctuations are not expected to be overwhelmingly large.

In the ordered phase, on the other hand, the situation is different. The spectrum contains $O(N^2)$ "Goldstone bosons". In the mean field approximation in addition it also has $O(N^2)$ massive particles — the analogs of δ particles in QCD. In the mean field the δ's are all stable. In reality of course, since there is nothing to prevent them from decaying into the Goldstone bosons, they are not at all stable. In fact, the further into the ordered phase one gets, the less stable they are due to the increase of the phase space available for decay. Thus we expect the mean field to be unreliable in the ordered phase. Asymptotically

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at very large M the mass of δ becomes much higher than the ultraviolet cutoff of the σ-model. For these values of M the massive modes decouple even within the mean field approximation, and one recovers the perturbative result. This decoupling, however, occurs extremely slowly. We have checked that the mean field formulae tend to the perturbative ones only when the mass of the would be stable massive states is about 1000 times the ultraviolet cutoff. This is, of course, unphysical, confirming our suspicion that the mean field approximation cannot be used on the ordered side of the transition.

The simplest option to calculate the contribution of the low momentum modes to the energy in the ordered phase is to use perturbation theory. The *raison d'etre* for this is the following. Perturbation theory is certainly appropriate for large enough values of M, where the expectation value of the U field is of order unity. On the other hand, we know from the numerical studies of [18] that the phase transition in our σ-model is strongly first order. More specifically, according to [18] the transition occurs when the expectation value of U is greater than .5. We thus expect perturbation theory to be qualitatively reliable all the way down to the transition point. Since we have neglected previously many perturbative contributions of order g^2, in order to be consistent we have to limit ourselves to the leading order contribution only. Calculating with this precision the energy of the trial states for $M \geq M_c$ we get

$$\frac{\mathsf{E}(M)}{V} = \frac{N^2}{120\pi^2} M^4 \,. \tag{2.30}$$

Thus for $M \geq M_c$ the energy is a monotonically increasing function of M.

Eq. (2.29) is thus the result of the minimization of the energy over the whole range of the variational parameter M.

3. The variational ansatz for the density matrix

In order to extend the variational analysis to finite temperature we have to generalize our ansatz so that it includes mixed states. In scalar theories the gaussian approximation has a long history of applications at finite temperature [19, 20]. We generalize our ansatz along the same lines.

We start by considering the density matrices which in the field basis have gaussian matrix elements

$$\begin{aligned}\varrho[A, A'] = \exp\Bigg\{ -\frac{1}{2}\int_{x,y} & A_i^a(x) G_{ij}^{-1ab}(x,y) A_j^b(y) + \\ & + A_i'^a(x) G_{ij}^{-1ab}(x,y) A_j'^b(y) - 2A_i^a(x) H_{ij}^{ab}(x,y) A_j'^b(y) \Bigg\} \,.\end{aligned} \tag{3.1}$$

As before, we take the variational functions diagonal in both colour and Lorentz indices, and translationally invariant

$$\begin{aligned} G_{ij}^{-1ab}(x,y) &= \delta^{ab}\delta_{ij} G^{-1}(x-y) \,, \\ H_{ij}^{ab}(x,y) &= \delta^{ab}\delta_{ij} H(x-y) \,. \end{aligned} \tag{3.2}$$

Then

$$\tilde{\varrho}[A,A'] = \exp\left\{ -\frac{1}{2}\int_{x,y} \left(AG^{-1}A + A'G^{-1}A' - 2AHA'\right)\right\}. \tag{3.3}$$

Note that for $H = 0$ this density matrix represents a pure state, since it can be written in the form

$$\tilde{\varrho} = |\Psi[A]\rangle\langle\Psi[A]| \tag{3.4}$$

with $\Psi[A]$ a gaussian wave function. At nonzero H the density matrix is, however, mixed. The magnitude of H, therefore, determines the entropy of this trial density matrix.

We now make an additional simplification in our ansatz. First, we restrict the functions $G^{-1}(x)$ to the same functional form as at zero temperature, eq. (2.8). Further, we will take $H(k)$ to be small and nonvanishing only at low momenta

$$H(k) = \begin{cases} 0 & \text{if } k^2 > M^2 \\ H \ll M & \text{if } k^2 < M^2 \end{cases}. \tag{3.5}$$

The logic behind this choice of ansatz is the following. At finite temperature we expect $H(k)$ to be roughly proportional to the Bolzmann factor $\exp\{-\mathsf{E}(k)\beta\}$. In our ansatz, the role of one particle energy is played by the variational function $G^{-1}(k)$. We will be interested only in temperatures close to the phase transition, and those we anticipate to be small, $\mathsf{T_c} \leq M$. For those temperatures one particle modes with momenta $k \geq M$ are not populated, and we thus put $H(k) = 0$. For $k \leq M$ the Bolzmann factor is nonvanishing, but small. Further, it depends only very weakly on the value of the momentum. We will have, of course, to verify a posteriori that our assumptions about the smallness of $\mathsf{T_c}$ and H are justified. As we will see later, this turns out indeed to be the case with reasonable accuracy.

As before, we explicitly impose gauge invariance by projecting ϱ onto a gauge invariant sector

$$\varrho[A,A'] = \int \mathcal{D}U'\mathcal{D}U'' \exp\left\{ -\frac{1}{2}\int_{x,y} A^{U'}G^{-1}A^{U'} + A'^{U''}G^{-1}A'^{U''} - 2A^{U'}HA'^{U''}\right\}. \tag{3.6}$$

One of the group integrations in eq. (3.6) is redundant, since we will only calculate the quantities of the form $\mathrm{Tr}\,\varrho O$, with O being gauge invariant. We thus have the following ansatz for the density matrix

$$\varrho[A,A'] = \int \mathcal{D}U \exp\left\{ -\frac{1}{2}\int_{x} AG^{-1}A + A'^{U}G^{-1}A'^{U} - 2AHA'^{U}\right\}. \tag{3.7}$$

This expression is not explicitly normalized to unity. Nevertheless, we find it convenient to refer to it as density matrix while explicitly inserting a normalization factor whenever necessary. Thus the average of a gauge invariant operator $\mathcal{O}$ is given by

$$\begin{aligned} \langle\mathcal{O}\rangle_{A,U} &= Z^{-1}\,\mathrm{Tr}(\varrho\mathcal{O}) \qquad (3.8)\\ &= Z^{-1}\int \mathcal{D}U\mathcal{D}A\,\mathcal{O}(A,A')\exp\left\{ -\frac{1}{2}\int_{x} AG^{-1}A + A'^{U}G^{-1}A'^{U} - 2AHA'^{U}\right\}\Bigg|_{A'=A}, \end{aligned}$$

where Z is the normalization of the trial density matrix ϱ, i.e.

$$\begin{aligned} Z = \mathrm{Tr}\,\varrho &= \int \mathcal{D}U\mathcal{D}A \exp\left\{ -\frac{1}{2}\int_x AG^{-1}A + A^U G^{-1} A^U - 2AHA^U \right\} \\ &= \int \mathcal{D}U\mathcal{D}\tilde{A} \exp\left\{ -\frac{1}{2}\int_x \tilde{A}\Delta\tilde{A} + \lambda\left(G^{-1} - \omega\Delta^{-1}\omega^T\right)\lambda \right\} \end{aligned} \tag{3.9}$$

with

$$\tilde{A} = A + \lambda\omega\Delta^{-1}\,, \tag{3.10}$$

$$\Delta = 2G^{-1}\Big(1 - \frac{HG}{2}(S + S^T)\Big)\,, \tag{3.11}$$

$$\omega = \big(G^{-1}S - H\big)\,. \tag{3.12}$$

The $\tilde{A}$ integration can then be easily performed to yield

$$\mathrm{Tr}\,\varrho = \int \mathcal{D}U \exp\left\{ -\frac{1}{2}\lambda\left(G^{-1} - \omega\Delta^{-1}\omega^T\right)\lambda - \frac{3}{2}\,\mathrm{Tr}\ln\frac{\Delta}{2} \right\}. \tag{3.13}$$

We now adopt the same strategy for treating the high momentum modes of U as at $\mathsf{T} = 0$. Namely, they are integrated perturbatively to one loop accuracy. The result is the effective σ-model for the matrices U with momenta below M. The coupling constant g of this σ-model gets renormalized as before according to the one loop Yang-Mills β-function, and thus has to be understood as $g(M)$. Additionally, due to independence of H on momentum, for low momentum modes of U the function $H(x - y)$ is equivalent to $H\delta^3(x - y)$.

The final approximation has to do with the fact that H is assumed to be small. For arbitrarily large H the variational calculation is forbiddingly complicated even with all the above mentioned simplifications. This is because the gauge projection renders the calculation of entropy in the general case unfeasible. However, at small H we only need to calculate the leading term in entropy. This calculation can indeed be done, and is described in the next section. Since we are only calculating the leading order contribution in H, we only have to consider corrections to the σ-model action of first order in H. With this in mind, the normalization factor becomes

$$\mathrm{Tr}\,\varrho = \int \mathcal{D}U \exp\left\{ -\frac{1}{2}\lambda\left(\frac{G^{-1}}{2} + \frac{H}{4}\left(S + S^T\right)\right)\lambda + \frac{3}{4}HG\ \mathrm{tr}\left(S + S^T\right) \right\}. \tag{3.14}$$

4. The effective σ-model

The normalization Z can be interpreted as the generating functional for a theory defined by the action $\mathcal{S}(U)$

$$Z = \mathrm{Tr}\,\varrho = \int \mathcal{D}U e^{-\mathcal{S}(U)}\,, \tag{4.1}$$

where

$$\mathcal{S}(U) = \frac{M}{4}\lambda\lambda + \frac{1}{8}\lambda H\left(S + S^T\right)\lambda - \frac{1}{4\pi^2}HM^2\,\mathrm{tr}\,S\,. \tag{4.2}$$

We simplify this expression using

$$\lambda\lambda = \frac{2}{g^2}\,\mathrm{tr}\left(\partial U \partial U^\dagger\right), \tag{4.3}$$

$$\lambda S^T \lambda = \lambda S \lambda = -\frac{1}{2g^2}\,\mathrm{tr}\Big[\left(U^\dagger \partial U - \partial U^\dagger U\right)\left(\partial U U^\dagger - U \partial U^\dagger\right)\Big], \tag{4.4}$$

$$\mathrm{tr}\, S = \mathrm{tr}\, S^T = \mathrm{tr}\, U^\dagger\, \mathrm{tr}\, U - 1\,. \tag{4.5}$$

Inserting these into the action we get

$$\begin{aligned}\mathcal{S}(U) = &\frac{M}{2g^2}\,\mathrm{tr}\left(\partial U \partial U^\dagger\right) - \frac{H}{8g^2}\,\mathrm{tr}\Big[\left(U^\dagger \partial U - \partial U^\dagger U\right)\left(\partial U U^\dagger - U\partial U^\dagger\right)\Big] - \\ &- \frac{1}{4\pi^2} H M^2\,\mathrm{tr}\, U^\dagger\, \mathrm{tr}\, U\,,\end{aligned} \tag{4.6}$$

where U independent pieces have been dropped.

The matrix U plays the same role as Polyakov's loop P at finite temperature - the functional integration over U projects out the physical subspace of the large Hilbert space on which the hamiltonian of gluodynamics is defined. The effective σ-model eq. (4.6) therefore is a close analog of the effective theory for the low momentum modes of the Polyakov loop variable. Its status and applicability region are different from the usual perturbative effective actions, see e.g. [21]. The standard effective action is calculated in perturbation theory and is valid at high temperature. Our effective action eq. (4.6) depends on the variational parameters M and H, and in a sense is a variational effective action. Also due to our restrictions to small values of H, a priori we do not expect it to be valid at high temperatures but, it rather, should represent correctly the physics in the phase transition region.

Another important difference is that our effective σ-model does not have the local gauge invariance $U(x) \to V^\dagger(x)U(x)V(x)$ which is usually associated with the effective action for the Polyakov loop. The reason for this is that our setup is different from that of the standard finite temperature calculation. The way this gauge invariance usually appears is the following. Consider the calculation of any gauge invariant observable

$$\langle \mathcal{O} \rangle = \int \mathcal{D}U\, \mathrm{Tr}[\exp\{-\beta \mathsf{H}\}\mathcal{O} g(U)]\,, \tag{4.7}$$

where $g(U)$ is the quantum mechanical operator of the gauge transformation represented by the matrix U. This expression for fixed U can be compared to the same expression but with U gauge transformed

$$\mathrm{Tr}\Big[\exp\{-\beta\mathsf{H}\}\mathcal{O} g\left(V^\dagger U V\right)\Big] = \mathrm{Tr}\Big[\exp\{-\beta\mathsf{H}\}\mathcal{O} g\left(V^\dagger\right) g(U) g(V)\Big] = \mathrm{Tr}[\exp\{-\beta\mathsf{H}\}\mathcal{O} g(U)]\,. \tag{4.8}$$

The last equality here follows from the fact that both $\mathcal{O}$ and $\exp\{\beta\mathsf{H}\}$ are gauge invariant, and thus the operator $g(V^\dagger)$ can be commuted all the way to the left. The only effect of the transformation is then to change the basis over which the trace is being taken, which obviously leaves the trace invariant.

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Our setup is somewhat different. Expectation values are calculated as

$$\int \mathcal{D}U \, \mathrm{Tr}[\tilde{\varrho} g(U) \mathcal{O}] \tag{4.9}$$

with $\tilde{\varrho}$ defined in eq. (3.3). This expression is altogether gauge invariant, since the integral over U correctly projects only the contribution of gauge singlet states. However the operator $\tilde{\varrho}$ is not itself explicitly gauge invariant. For that reason the gauge transformation operator $g(V^\dagger)$ cannot be commuted through it, and thus

$$\mathrm{Tr}\left[\tilde{\varrho} \mathcal{O} g\left(V^\dagger U V\right)\right] \neq \mathrm{Tr}[\tilde{\varrho} \mathcal{O} g(U)] \tag{4.10}$$

even for gauge invariant operators $\mathcal{O}$. This manifests itself as absence of local gauge invariance in the action of the effective σ-model, eq. (4.6).

Nevertheless, we stress again that since the integration over the $\mathrm{SU}(N)$ valued field U projects out the physical Hilbert space, its meaning in this sence is the same as that of the Polyakov loop.

5. The calculation of the free energy

To find the best variational density matrix we have to minimize the free energy with respect to the variational parameters M and H. The Helmholtz free energy F of the density matrix ϱ is given by

$$\mathsf{F} = \langle \mathsf{H} \rangle - \mathsf{TS}\,, \tag{5.1}$$

where H is the standard Yang-Mills hamiltonian eq. (2.15), S is the entropy, and T is the temperature.

Thus

$$\mathsf{F} = \frac{1}{2}\left(\mathrm{Tr}\left(E^2 \varrho\right) + \mathrm{Tr}\left(B^2 \varrho\right)\right) + \mathsf{T} \cdot \mathrm{Tr}(\varrho \ln \varrho)\,. \tag{5.2}$$

First of all we need to perform the integration over the gauge fields, and reduce this expression to the average of a U-dependent operator in the effective σ-model. In fact, as we shall see soon, to leading order in H the only nontrivial calculation we need to perform is that of the entropy.

We will calculate the entropy up to the first nontrivial order in H. As we now show, the leading term at small H is $O(H \ln H)$.

Let us denote by ϱ_0 the density matrix of the pure state with $H = 0$:

$$\varrho_0 = |0\rangle\langle 0|\,. \tag{5.3}$$

Here $|0\rangle$ does not denote necessarily the actual ground state, but rather a projected gaussian state with arbitrary M. Now, since the matrix elements of the density matrix can be expanded in powers of H, to leading order we can write

$$\varrho = \varrho_0 + \delta\varrho\,, \tag{5.4}$$

where $\delta\varrho$ is $O(H)$.

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Imagine that we have diagonalized ϱ. It will have one large eigenvalue $\alpha_0 = 1 - O(H)$, which corresponds to the eigenstate

$$|0'\rangle = |0\rangle + O(H)\,. \tag{5.5}$$

All the rest of the eigenvalues α_i are at most $O(H)$. Then the entropy can be written as

$$\mathsf{S} = -\operatorname{Tr}(\varrho \ln \varrho) = -\alpha_0 \ln \alpha_0 - \sum_{i=1}^{\infty} \alpha_i \ln \alpha_i\,. \tag{5.6}$$

The second term is $O(H \ln H)$, and it is the coefficient of this term that we will now calculate. Neglecting $O(H)$ corrections, we can substitute $\alpha_i = H/M$ under the logarithm. Thus to leading logarithmic order

$$\mathsf{S} = -\sum_i \alpha_i \ln H/M\,. \tag{5.7}$$

So that all we have to calculate is $\sum_i \alpha_i$. Let

$$|0'\rangle = |0\rangle + H|x\rangle\,. \tag{5.8}$$

Then

$$\varrho = \alpha_0 |0'\rangle\langle 0'| + \sum_{i=1}^{\infty} \alpha_i |x_i\rangle\langle x_i| \tag{5.9}$$

with

$$\langle x_i|0'\rangle = 0\,. \tag{5.10}$$

Note that $\langle 0|x\rangle \neq 0$, but

$$\langle 0|x\rangle + \langle x|0\rangle = 0\,, \tag{5.11}$$

since $|0'\rangle$ has to be normalized at $O(H)$. Also

$$\langle x_i|0\rangle + H\langle x_i|x\rangle = 0\,. \tag{5.12}$$

Thus the overlap $\langle x_i|0\rangle$ is $O(H)$, and we have

$$\varrho = \alpha_0 |0\rangle\langle 0| + H\Big(|0\rangle\langle x| + |x\rangle\langle 0|\Big) + \alpha_i |x_i\rangle\langle x_i|\,. \tag{5.13}$$

Multiplying this by ϱ_0 we get

$$\begin{aligned}\varrho_0 \cdot \varrho &= \alpha_0 \cdot \varrho_0 + H|0\rangle\langle x| + H\langle 0|x\rangle|0\rangle\langle 0|\,,\\ \varrho \cdot \varrho_0 &= \alpha_0 \cdot \varrho_0 + H|x\rangle\langle 0| + H\langle x|0\rangle|0\rangle\langle 0|\,.\end{aligned} \tag{5.14}$$

Thus,

$$\varrho_0 \cdot \varrho + \varrho \cdot \varrho_0 - \varrho = \alpha_0 \cdot \varrho_0 - \alpha_i |x_i\rangle\langle x_i|\,. \tag{5.15}$$

Multiplying by ϱ_0 again, we get rid of $|x_i\rangle\langle x_i|$ to $O(H)$

$$\alpha_0 \cdot \varrho_0 = \varrho_0 \cdot \varrho + \varrho_0 \cdot \varrho \cdot \varrho_0 - \varrho_0 \cdot \varrho = \varrho_0 \cdot \varrho \cdot \varrho_0\,. \tag{5.16}$$

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Then,

$$\alpha_0 = \mathrm{Tr}(\varrho_0 \cdot \varrho)\,. \tag{5.17}$$

Since $\mathrm{Tr}\,\varrho = 1$ we have

$$\sum_i \alpha_i = 1 - \alpha_0 = \mathrm{Tr}(\varrho_0(1-\varrho)) \tag{5.18}$$

which, inserted into eq. (5.6), gives

$$\mathsf{S} = -\left(1 - \mathrm{Tr}(\varrho_0 \cdot \varrho)\right) \ln H/M\,. \tag{5.19}$$

The derivation has been given for the normalized density matrices ϱ_0 and ϱ. In terms of our gaussian matrices we should restore the normalization factors Z and Z_0, so that finally we have

$$\mathsf{S} = \left(\frac{\mathrm{Tr}(\varrho_0 \cdot \varrho)}{\mathrm{Tr}\,\varrho_0 \cdot \mathrm{Tr}\,\varrho} - 1\right) \ln H/M\,. \tag{5.20}$$

It is easy to check that to $O(H)$

$$\mathrm{Tr}(\varrho_0 \cdot \varrho) = (\mathrm{Tr}\,\varrho_0)^2\,. \tag{5.21}$$

Then

$$\mathsf{S} = \left(\frac{\mathrm{Tr}\,\varrho_0}{\mathrm{Tr}\,\varrho} - 1\right) \ln H/M\,. \tag{5.22}$$

From eq. (3.14), it is clear that

$$\mathrm{Tr}\,\varrho = \left[1 + H\left(\frac{1}{4\pi^2}M^2\,\mathrm{tr}\,S - \frac{1}{4}\lambda S \lambda\right)\right] \cdot \mathrm{Tr}\,\varrho_0\,. \tag{5.23}$$

Using eqs. (4.4), (4.5) we finally get

$$\mathsf{S} = -\left[\left\langle \frac{1}{8g^2}\,\mathrm{tr}\left(U^\dagger\partial U - \partial U^\dagger U\right)\left(\partial U U^\dagger - U\partial U^\dagger\right) + \frac{1}{4\pi^2}M^2\left(\mathrm{tr}\,U^\dagger\,\mathrm{tr}\,U - 1\right)\right\rangle_U\right] H \ln H/M\,. \tag{5.24}$$

To leading order in H, the averaging over U in this expression has to be performed with the σ-model action at vanishing H.

Expression eq. (5.24) has the following striking property. For $M < M_c$ it vanishes identically. The reason is very simple. The first term in eq. (5.24) is the product of the left handed SU(N) current and the right handed SU(N) current in the σ-model. Thus it transforms as an adjoint representation under each one of the SU(N) factors of the $SU_L(N) \otimes SU_R(N)$ transformation. The same is also true for the second term in eq. (5.24). The σ-model action itself at $H = 0$ obviously is invariant under the whole $SU_L(N) \otimes SU_R(N)$ group. Now, at $M < M_c$, the symmetry group is not spontaneously broken, and thus any operator which is not a scalar has a vanishing expectation value. It follows immediately that the entropy has an $O(H \ln H/M)$ contribution only for $M > M_c$, when the $SU_L(N) \otimes SU_R(N)$ group is spontaneously broken down to $SU_V(N)$.

This observation makes our task considerably simpler. Since for $M < M_c$ the entropy is zero, we do not have to consider at all the disordered phase of the effective σ-model. In this

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disordered phase the free energy coincides with energy, and thus the calculation is identical to the calculation at zero temperature. There is one slight subtlety here. We expect of course that since the energy alone must have a minimum on a pure state, the nonvanishing H should give always a positive contribution to the energy. Thus in the absence of the entropy contribution, H should vanish in the minimal energy state. Nevertheless since in [9] the energy at $\mathsf{T} = 0$ was only minimized with respect to M, we have to check this property explicitly. We have indeed performed this check and found that for small H the derivative of the expectation value of the energy with respect to H is strictly positive.

Thus we only need to consider the effective σ-model in the ordered phase. As at $\mathsf{T} = 0$ we perform the calculations in the ordered phase to leading order in α_s. Since there are no $O(H \ln H/M)$ corrections to energy at this order, the result for the energy in the disordered phase is identical to the result at zero temperature, eq. (2.30). Thus our expression for the free energy is

$$\begin{aligned} \mathsf{F} = \frac{N^2}{120\pi^2} M^4 + \mathsf{T} \Bigg(\Big\langle \frac{1}{8g^2} \operatorname{tr} \left(U^\dagger \partial U - \partial U^\dagger U \right) \left(\partial U U^\dagger - U \partial U^\dagger \right) + \\ + \frac{1}{4\pi^2} M^2 \left(\operatorname{tr} U^\dagger \operatorname{tr} U - 1 \right) \Big\rangle_U \Bigg) H \ln H/M \,. \end{aligned} \tag{5.25}$$

We now average over U in the leading order perturbation theory.

6. The σ-model perturbation theory — minimization of the free energy and the Debye mass

We parameterize the U matrices as

$$U = \exp\left\{ -\frac{i}{2} \phi^a \tau^a \right\}. \tag{6.1}$$

Although we only need the leading order, it is instructive to check that the order g^2 term in the expansion is indeed small. To this order we have

$$U \simeq \left(1 - \frac{i}{2} \phi^a \tau^a - \frac{1}{8} \phi^a \phi^b \tau^a \tau^b + \frac{i}{48} \phi^a \phi^b \phi^c \tau^a \tau^b \tau^c \right). \tag{6.2}$$

So that the σ-model action becomes

$$\begin{aligned} \mathcal{S} &= \frac{M}{2g^2} \operatorname{tr} \left(\partial U \partial U^\dagger \right) \\ &= \frac{M}{4g^2} \partial\phi \partial\phi + \frac{M}{192 g^2} (\partial\phi^a)(\partial\phi^c) \phi^b \phi^d \operatorname{tr} \left[\tau^a \tau^b \tau^c \tau^d - \tau^a \tau^c \tau^b \tau^d \right]. \end{aligned} \tag{6.3}$$

The propagator of the phase field ϕ is thus

$$\langle \phi^a \phi^b \rangle = \frac{2g^2}{Mk^2} \delta^{ab} \,. \tag{6.4}$$

To get the idea of the quality of this perturbative expansion we can calculate for example $\langle \mathcal{S} \rangle$. In this calculation one has to take into account the fact that the measure in the path integral over the phase ϕ^a is not the simple $\mathcal{D}\phi$, but rather the group invariant U(N) measure μ . To first order in g^2 it is

$$\mu = \mathcal{D}\phi^a \exp\left\{\frac{M^3 N}{144\pi^2}\int d^3x \phi^2(x)\right\} . \tag{6.5}$$

Taking this into account we find that $\langle \mathcal{S}\rangle$ gets no correction of order g^2. We thus feel confident that the use of the perturbation theory in the ordered phase of the σ-model is an admissible approximation. In the following we will only keep leading order expressions.

Calculating to leading order the entropy eq. (5.24) and keeping only the $O(N^2)$ terms we find

$$\langle \mathsf{S}\rangle = -\frac{N^2}{6\pi^2}M^2 H \ln\frac{H}{M} . \tag{6.6}$$

Introducing the dimensionless quantity

$$h = \frac{H}{M} \tag{6.7}$$

we can write the expression for the free energy as

$$\begin{aligned} \mathsf{F} &= \langle \mathsf{H}\rangle - \mathsf{T}\langle \mathsf{S}\rangle \\ &= \frac{N^2}{120\pi^2}M^4 + \mathsf{T}\frac{N^2}{6\pi^2}M^3 h \ln h . \end{aligned} \tag{6.8}$$

We now have to minimize this expression with respect to h and M. It is convenient to first perform the minimization with respect to h at fixed M. This obviously gives

$$\frac{\partial \mathsf{F}}{\partial h} = 0 \quad \rightarrow \quad h = \frac{1}{e} . \tag{6.9}$$

Thus as a function of M only, the free energy becomes

$$\mathsf{F} = \frac{N^2}{120\pi^2}M^4 - \frac{\mathsf{T}}{e}\frac{N^2}{6\pi^2}M^3 . \tag{6.10}$$

Now minimizing with respect to M we find

$$\frac{\partial \mathsf{F}}{\partial M} = 0 \quad \rightarrow \quad M = \frac{15\mathsf{T}}{e} . \tag{6.11}$$

Thus for $M \geq M_c$ the free energy of the best variational density matrix as a function of temperature is

$$\mathsf{F}_{M\geq M_c} = -\frac{N^2}{360\pi^2}\left(\frac{15\mathsf{T}}{e}\right)^4 . \tag{6.12}$$

We now have to compare this value with the free energy for $M \leq M_c$. As we have discussed above, this is given by the expectation value of the hamiltonian alone, and is minimized at $M = M_c$. Its value is

$$\mathsf{F}_{M\leq M_c} = -\frac{N^2}{30\pi^2}M_c^4 . \tag{6.13}$$

Comparing the two expressions we find

$$\mathsf{T_c} = \frac{12^{\frac{1}{4}} e}{15} M_c\,. \tag{6.14}$$

Using the value of M_c from eq. (2.29) we have

$$\mathsf{T_c} = 450\,\mathrm{MeV}\,. \tag{6.15}$$

For $\mathsf{T} \leq \mathsf{T_c}$ the free energy is minimized in the variational state with $M = M_c$. In our approximation this state is the same as at zero temperature. Its entropy vanishes, and the effective σ-model is in the disordered phase. The Polyakov loop vanishes, $\langle U\rangle = 0$ and according to the standard wisdom this is a confining state.

For $\mathsf{T} \geq \mathsf{T_c}$ the best variational state is very different. The entropy of this state is nonzero,

$$\mathsf{S} = \frac{N^2}{6\pi^2 e}\left(\frac{15\mathsf{T}}{e}\right)^3. \tag{6.16}$$

The Polyakov loop is nonzero $\langle U\rangle \neq 0$ and thus the high temperature density matrix describes a deconfined phase.

Finally, we note that in the deconfined phase our best variational density matrix has a nonvanishing "electric screening" or "Debye" mass. The Debye mass is conveniently defined as the "mass" of the phase of the Polyakov loop. This mass is nonvanishing in our calculation for the following reason. As long as $H = 0$, the effective σ-model action has a global $SU_L(N)\otimes SU_R(N)$ symmetry. Thus in the ordered phase of the σ model the phases ϕ^a are massless. However, as discusses above, the terms of order H in eq. (4.6) break this symmetry explicitly down to the diagonal $SU_V(N)$. As a result the would be "Goldstone" phases ϕ^a acquire mass. To calculate this mass it is convenient first to note that to $O(g^2)$

$$\mathrm{tr}(U^\dagger\partial U - \partial U^\dagger U)(\partial U U^\dagger - U\partial U^\dagger) = -4\,\mathrm{tr}\,\partial U^\dagger \partial U - \frac{1}{4}\phi^a\phi^c\partial\phi^b\partial\phi^d\,\mathrm{tr}(\tau^a\tau^b\tau^c\tau^d - \tau^a\tau^c\tau^b\tau^d). \tag{6.17}$$

The contribution of the $SU_L(N)\otimes SU_R(N)$ term to the mass cancels against the contribution of the measure eq. (6.5). Using eqs. (4.6), (6.17) we then find to $O(g^2)$ and to leading order in H

$$M_D^2 = \frac{4}{3\pi}\alpha_s(M) N M H\,. \tag{6.18}$$

As a function of temperature we have

$$M_D^2 = \alpha_s\left(\frac{15}{e}T\right) N \frac{300}{\pi e^3}\mathsf{T}^2\,. \tag{6.19}$$

7. Discussion

Let us summarize the results of our variational calculation. We find the phase transition at a temperature of about $\mathsf{T_c} \simeq 450\,\mathrm{MeV}$. The transition is strongly first order at large-N. The latent heat is $\Delta E = \frac{N^2}{90\pi^2}(\frac{15\mathsf{T}}{e})^4$. Below the transition the entropy is zero, the best variational state is the same as at zero temperature, and the average value of the Polyakov loop is zero. Above the transition, the entropy is nonzero and proportional to the number of "coloured" degrees of freedom, $\mathsf{S} \propto N^2$. The average value of the Polyakov loop is nonzero and the phase is deconfined.

It is quite interesting that at high temperature our formulae numerically are quite close to the predictions of free gluon plasma. In particular, our value for the free energy, eq. (6.12), should be compared to the free gluon plasma expression

$$\mathsf{F}_{\text{free}} = -\frac{N^2\pi^2}{45}\mathsf{T}^4 . \tag{7.1}$$

The ratio between the two is

$$\frac{\mathsf{F}_{\text{free}}}{\mathsf{F}_{\text{var}}} \simeq 0.85 . \tag{7.2}$$

The ratio of the entropies is the same.

Interestingly we get the same ratio comparing our value for the Debye mass eq. (6.19) with the leading order perturbative one, $M^2_{pert} = \frac{4\pi}{3}\alpha_s N T^2$,

$$\frac{M^2_{\text{pert}}}{M^2_D} \simeq 0.85 . \tag{7.3}$$

The pressure approaches its asymptotic value according to the simple formula

$$\frac{\mathsf{P}(\mathsf{T})}{\mathsf{P}_{\text{asympt}}} = 1 - \frac{{\mathsf{T}_\mathsf{c}}^4}{\mathsf{T}^4} . \tag{7.4}$$

Here the asymptotic value of the pressure $\mathsf{P}_{\text{asympt}}$ is given by eq. (6.12). The pressure $\mathsf{P}(\mathsf{T})$ is given by the difference between eq. (6.12) and the value of the free energy at zero temperature, which coincides with expression eq. (6.13).

One has to take the comparison eqs. (7.2), (7.3) with a grain of salt. As explained above, our calculations were performed assuming small H. A priori we expect that this restriction should confine us to not too large temperatures. Interestingly, however, the minimization of the free energy resulted in the value $H/M = 1/e$ independently of temperature. Thus, we feel ourselves justified to consider the comparison eq. (7.2) as meaningful.

The main features of these results are indeed what we expect from the deconfinement phase transition on general grounds. It is extremely gratifying that a simple minded calculation such as ours does qualitatively so well in such a complicated problem. It therefore appears that the projection of the trial density matrix on the gauge invariant Hilbert space is, just like at zero temperature, the crucial feature that dictates most if not all the important aspects of the low energy and low temperature physics. In the context of the present calculation the most important effect of the gauge projection is obviously vanishing of the entropy in the low temperature phase. We stress that this feature was not at all built into our initial ansatz, but followed naturally and unavoidably in the disordered phase of the effective σ-model.

Quantitatively, our calculation of course should be taken for what it is — an approximate implementation of the variational principle. As with any variational calculation, the range of validity of this calculation is not sharply defined. Even within the variational framework we had to resort to additional approximations. The most severe simplifications that we had to impose are the perturbation theory in the ordered phase of the σ-model and the assumption of smallness of H. The projection over the gauge group, which as we saw is so physically important, is what makes the calculational task difficult and forces us to make these approximations.

The assumptions of smallness of g and of smallness of H affect different aspects of our result. In particular, in the leading order of the perturbation theory the expectation value of the Polyakov loop U is equal to unity. The actual value of U on the ordered side of the transition according to [18] is close to one half. Thus our perturbative calculation is rather more reliable somewhat further away from the transition. The closer to the transition we get, the more important higher order corrections in g^2 become. Thus to properly describe the transition region of QCD we need to improve our calculational method in the vicinity of the transition in the σ-model. In line with this we expect that the estimate for the critical temperature we obtained here is somewhat higher than we would get, had we treated the σ-model more accurately in the transition region. This is consistent with the fact that our result for $\mathsf{T_c}$ is by about 50% higher than the lattice value of 270 MeV.

The smallness of h is quite important in a different way. The value of $h = 1/e$ that we obtain is in fact a reasonably small number, so omitting the corrections in powers of h is fairly safe. On the other hand, the terms linear in h but not enhanced by $\ln h$, which we have ignored in the present calculation, have to be accounted for more carefully. With the value of h that we obtain, these terms are not suppressed in any obvious way. Once these terms are taken care of our calculation can be extended to high temperatures. In particular, the 15% discrepancy between our asymptotic results and the expectations for the free gluon plasma eqs. (7.2), (7.3) must be due to our neglect of these terms. In a way we were fortunate to obtain the result so close to the correct asymptotic one at high T.

We believe that with some effort both these limitations can be overcome, at least to some extent. It should be possible to treat the σ-model action in a better way, perhaps along the lines of a continuum version of [18]. As for terms linear in h, those should be accessible in perturbative expansion in h, once the nonanalytic $O(h \ln h)$ terms have been understood. We plan to continue investigations along these lines and hope to improve on the quality of our results.

Notwithstanding this critique, we are very much encouraged by the results of this exploratory investigation. We believe that a reasonably simple improvement on the calculational methods has a good chance to significantly improve the results and bring them into a better quantitative agreement with the lattice data.

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